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Condorcet domains of tiling type

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ABSTRACT

A Condorcet domain (CD) is a collection of linear orders on a set of candidates satisfying the following property: for any choice of preferences of voters from this collection, a simple majority rule does not yield cycles. We propose a method of constructing "large" CDs by use of rhombus tiling diagrams and explain that this method unifies several constructions of CDs known earlier. Finally, we show that three conjectures on the maximal sizes of those CDs are, in fact, equivalent and provide a counterexample to them.

1. Introduction

In the social choice theory, a *Condorcet domain* (further abbreviated as a CD) is a collection of linear orders on a finite set of candidates (alternatives) such that if the voters choose their preferences to be linear orders belonging to this collection, then a simple majority rule does not yield cycles. For a survey, see, e.g., [15]. A challenging problem in the field is to construct CDs of "large" size. Several interesting methods based on different ideas have been proposed in the literature.

Abello [1] constructed CDs by a method of completing a maximal chain in the *Bruhat lattice*. (For maximal chains in the Bruhat lattice and their applications in combinatorics, see also [11].) Chameni-Nembua [3] proved that covering distributive sublattices in the Bruhat lattice are CDs. Fishburn [8] constructed CDs in the form of "alternating schemes", by using a clever combination of so-called "never conditions". An alternating scheme of this sort is a representative of an important class of CDs which we call *peak–pit* domains. Galambos and Reiner [9] developed an approach using the second Bruhat order. However, each of those methods (which are briefly reviewed in Appendix to this paper) is rather indirect, and it may take some efforts to see that the objects it generates are "good CDs" indeed.

In this paper, we construct a class of inclusion-wise maximal, or *complete*, CDs by use of known planar graphical diagrams called *rhombus tilings*. Our construction and proofs are rather transparent and the obtained CDs admit a good visualization. It should be noted that the obtained class of CDs is essentially the same as each of the above-mentioned classes.[1] We show that any peak–pit domain is a subdomain of a rhombus tiling CD (in Theorem 4). As a consequence, we obtain that three conjectures posed, respectively, by Fishburn, Monjardet, and Galambos and Reiner turn out to be equivalent. Finally, a simple example that we construct disproves these conjectures.

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1 The coincidence of the CD classes proposed by Abello and, Galambos and Reiner was established in [9].

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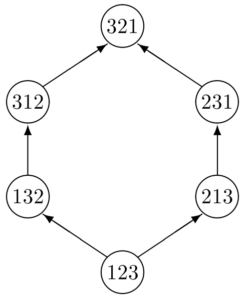

Fig. 1. The Bruhat digraph for $n = 3$.

2. Linear orders and the Bruhat poset

Let X be a finite set whose elements are interpreted as *alternatives*. A *linear order* on X is a complete transitive binary relation $<$ on X. It ranges the elements of X, and we can encode a linear order $x_1 < \cdots < x_n$ on X (where $n = |X|$) by the word $x_1 \cdots x_n$, regarding x_1 as the least (or worst) alternative, x_2 as the next alternative, and so on; then x_n is the greatest (or best) alternative. The set of linear orders on X is denoted by $\mathcal{L}(X)$. If $Y \subset X$, we have a natural restriction map $\mathcal{L}(X) \to \mathcal{L}(Y)$.

In what follows, the ground set X is identified with the set $[n]$ of integers $1, \ldots, n$. We usually use Greek symbols, say, σ, for linear orders on $[n]$, and write $i <_\sigma j$ rather than $i\sigma j$. The linear order $1 < 2 < \cdots < n$ is denoted by α, and the reversed order $n < (n-1) < \cdots < 1$ by ω.

Let $\Omega = \{(i, j): i, j \in [n], i < j\}$. For a linear order σ, a pair $(i, j) \in \Omega$ is called an *inversion* (w.r.t. α) if $j <_\sigma i$. The set of inversions for σ is denoted by $\mathrm{Inv}(\sigma)$. In particular, $\mathrm{Inv}(\alpha) = \emptyset$ and $\mathrm{Inv}(\omega) = \Omega$.

Definitions. For linear orders $\sigma, \tau \in \mathcal{L} = \mathcal{L}([n])$, we write $\sigma \ll \tau$ if $\mathrm{Inv}(\sigma) \subseteq \mathrm{Inv}(\tau)$. The relation $\ll$ on $\mathcal{L}$ is called the *weak Bruhat order*, and the partially ordered set $(\mathcal{L}, \ll)$ is called the *Bruhat poset*. A linear order τ *covers* a linear order σ if $\mathrm{Inv}(\tau)$ equals $\mathrm{Inv}(\sigma)$ plus exactly one inversion (this is known to agree with the notion of covering in a poset). The *Bruhat digraph* is formed by drawing a directed edge from σ to τ if and only if τ covers σ, and the underlying undirected graph is called the *Bruhat graph*.

Clearly, α and ω are the minimal and maximal elements of the Bruhat poset. It is known that this poset is a lattice. Also $(\mathcal{L}, \ll)$ is the transitive closure of the Bruhat digraph. For $n = 3$ this digraph is drawn in Fig. 1.

3. Condorcet domains

Let $\mathcal{D} \subseteq \mathcal{L}([n])$. We say that $\mathcal{D}$ is *cyclic* if there exist three alternatives i, j, k and three linear orders in $\mathcal{D}$ whose restrictions to $\{i, j, k\}$ are either of the form $\{ijk, jki, kij\}$ or $\{kji, jik, ikj\}$. An acyclic set $\mathcal{D}$ of linear orders is called a *Condorcet domain* (CD). Such domains are important since they admit aggregations (see, e.g., [15]).

More precisely, consider a mapping $\nu : \mathcal{D} \to \mathbb{Z}_+$ (called a $\mathcal{D}$*-opinion*), where $\nu(\sigma)$ is interpreted as the number of voters that pick a linear order σ. Then $|\nu| = \sum_{\sigma\in\mathcal{D}} \nu(\sigma)$ is the total number of voters. The "social preference" is defined to be the binary relation $sm(\nu)$ on $[n]$ constructed by the majority rule: $i\, sm(\nu) j \iff$ the number of voters which prefer i to j in their chosen linear orders is strictly more than those having the opposite preference. When the relation $sm(\nu)$ has no cycle for every $\mathcal{D}$-opinion ν, the set $\mathcal{D}$ is just a CD. (Indeed, it suffices to consider only $\mathcal{D}$-opinions where the total number of voters is odd (cf. [15]). Then the relation $sm(\nu)$ is complete, and the acyclicity of $\mathcal{D}$ implies that $sm(\nu)$ is a linear order on $[n]$. Conversely, if $\mathcal{D}$ is cyclic, then there exists a $\mathcal{D}$-opinion yielding a cycle in the "social preference".)

In the rest of this paper, we consider only domains $\mathcal{D} \subset \mathcal{L}([n])$ containing both distinguished orders α and ω (this, in fact, matches considerations in [1,3,8,9,15]). We say that $\mathcal{D}$ is *complete* if it is inclusion-wise maximal, i.e. adding to $\mathcal{D}$, any new linear order would violate the acyclicity.

One can check that in case $n = 3$, there are exactly four complete CDs. These are as follows.

(a) The set of four orders 123, 132, 312 and 321. These orders are characterized by the property that the alternative 2 is never the worst. We call this CD the *peak domain* (for $n = 3$) and denote it as $\mathcal{D}_3(\cap)$.
(b) The set of orders 123, 213, 231 and 321. In these orders, the alternative 2 is never the best. This CD is called the *pit domain* and denoted by $\mathcal{D}_3(\cup)$.
(c) The set $\{123, 213, 312, 321\}$. Here the alternative 3 is never the middle. We denote this domain by $\mathcal{D}_3(\to)$.
(d) The set $\{123, 132, 231, 321\}$, denoted by $\mathcal{D}_3(\leftarrow)$. Here the alternative 1 is never the middle.

A *casting* is meant to be a mapping c of the set $\binom{[n]}{3}$ of triples ijk $(i < j < k)$ to $\{\cap, \cup, \to, \leftarrow\}$. For a casting c, we define $\mathcal{D}(c)$ to be the set of linear orders $\sigma \in \mathcal{L}([n])$, whose restrictions to each triple ijk (further denoted as $\sigma|_{ijk}$) belongs to $\mathcal{D}_3(c(ijk))$. The following assertions are immediate.

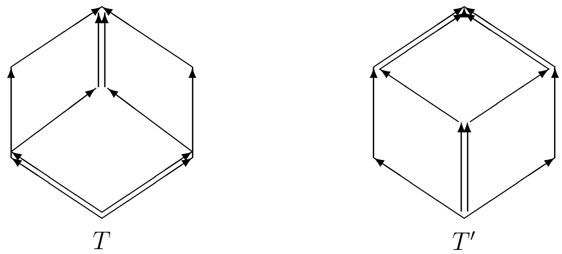

Fig. 2. Two tilings of the zonogon Z_3.

Proposition 1. (i) *For any casting c, $\mathcal{D}(c)$ is a Condorcet domain.*
(ii) *Any Condorcet domain is contained in a set $\mathcal{D}(c)$, where c is a casting.*

Note that a casually chosen casting may produce a small and/or non-complete CD. As Fishburn writes in [8]: "…it is far from obvious how the restrictions should be selected jointly to produce a large acyclic set". In the next section, we describe and examine a simple geometric construction generating a representable class of complete CDs.

4. Rhombus tilings and related CDs

The complete CDs that we are going to introduce one-to-one correspond to certain geometric arrangements on the plane, called rhombus tilings. In this section, we recall this notion, review basic properties of tilings needed to us, and finally we establish some facts about related CDs.

A. In the upper half-plane $\mathbb{R}\times\mathbb{R}_{>0}$, we fix n vectors $\xi_1,\ldots,\xi_n$ going in this order clockwise around $(0,0)$ and having the same length. The sum of segments $[0,\xi_i]$, $i=1,\ldots,n$, forms a *zonogon*, denoted by $Z=Z_n$. This is the center-symmetric $2n$-gon formed by the points $\sum_i a_i\xi_i$ over all $0\le a_i\le 1$. Two vertices of the zonogon are distinguished: the *bottom* vertex $b(Z):=(0,0)$ and the *top* vertex $t(Z):=\xi_1+\cdots+\xi_n$. A rhombus congruent to the sum of two segments $[0,\xi_i]$ and $[0,\xi_j]$, where $1\le i<j\le n$, is called an *ij-tile*, or simply a *tile*.

A *rhombus tiling* (or simply a *tiling*) is a subdivision T of the zonogon into a set of tiles satisfying the following condition: if two tiles intersect, then their intersection consists of a single vertex or a single (closed) edge. The set of tiles of T is denoted by $\mathrm{Rho}(T)$. Figs. 2 and 4 illustrate examples of rhombus tilings.

We associate to a tiling T the planar directed graph $G_T=(V_T,E_T)$ whose vertices and edges are those occurring in the tiles and the edges are oriented upward. The tiles of T are just the (inner two-dimensional closed) faces of G_T. An edge congruent to ξ_i is called an *i-edge*, or an edge of *color i*.

We will need two more definitions. First, since all edges of G_T are directed upward, this digraph is acyclic and any maximal directed path in it goes from $b(Z)$ to $t(Z)$. We call such a path a *snake* of T. In particular, the zonogon is bounded by two snakes, namely, those forming the left boundary $\mathrm{lbd}(Z)$ and the right boundary $\mathrm{rbd}(Z)$ of Z; note that the sequence of edge colors in the former (latter) gives the linear order α (resp. ω).

Second, for $i\in[n]$, we apply the term an *i-track* (borrowed from [12]) to a maximal alternating sequence $Q=(e_0,F_1,e_1,\ldots,F_k,e_k)$ formed by i-edges and different tiles, where e_{j-1},e_j are opposite edges of a tile F_j (other known names for Q are "de Bruijn line" [6], "dual i-path" and "i-stripe"). Note that the projections of $e_0,\ldots,e_k$ to a line orthogonal to ξ_i give a monotone sequence of points (since consecutive tiles in Q do not overlap). This implies that Q is not cyclic, is determined uniquely up to reversing, contains all i-edges of T, and connects the pair of i-edges on the boundary of the zonogon. We assume for definiteness that the i-track begins (with the edge e_0) on the left boundary of Z_n, and ends (with e_k) on the right boundary.

B. Next, we exhibit some properties of tilings. One important use of tracks consists in the following. When removing the i-track Q from the zonogon (i.e. removing the interiors of the edges and tiles of Q), we obtain two connected regions L_i, U_i such that: L_i (the *lower* region) contains the bottom vertex $b(Z)$ and U_i (the *upper* region) contains the top vertex $t(Z)$; the edges of G_T connecting these regions are exactly the i-edges $e_0,\ldots,e_k$ and these are directed from L_i to U_i; gluing L_i with U_i shifted by $-\xi_i$ produces the $(n-1)$-zonogon Z' generated by the vectors $\xi_1,\ldots,\xi_{i-1},\xi_{i+1},\ldots,\xi_n$. Moreover, removing from T the tiles of Q (and shifting those in U_i by $-\xi_i$) gives a rhombus tiling T' of Z'; we call T' the *reduction* of T by the color i and denote it as $T|_{[n]-i}$.

Using this operation and some other simple constructions and observations, one can demonstrate a number of rather elementary properties of tilings. Among these, the following nice properties of T are known.

Proposition 2. (i) *Any snake S intersects an i-track at exactly one i-edge. Therefore, S contains exactly n edges and the sequence of edge colors along S gives a linear order on $[n]$.*
(ii) *For any $1\le i<j\le n$, there is exactly one ij-tile in T. This yields a natural bijection $\psi:\mathrm{Rho}(T)\to\Omega$ (which maps an ij-tile to the pair $(i,j)\in\Omega$).*
(iii) *For a snake S of T, let σ be the linear order determined by S, and let $L(S)$, or $L(\sigma)$, denote the set of tiles of T lying on the left from S, i.e. those contained in the region bounded by S and $\mathrm{lbd}(Z)$. Then $\psi(L(\sigma))=\mathrm{Inv}(\sigma)$.*

(iv) *For a snake S, there exist two consecutive edges e, e' in S (where e precedes e') which have colors i and j, respectively, and belong to a tile $\rho \in \mathrm{Rho}(T)$ so that:* (a) *if $S \neq \mathrm{lbd}(Z)$, then $i > j$ and ρ lies on the left from S, and* (b) *if $S \neq \mathrm{rbd}(Z)$, then $i < j$ and ρ lies on the right from S.*

Remark. These facts (or somewhat close to them) were established in several works, possibly being formulated in different terms. See, e.g., [7,10,9,12,16]. Some authors (e.g., in [9]) prefer to operate in terms of so-called *commutation classes* of *pseudo-line arrangements* (visualizing *reduced words for permutations*; cf. [2]). Such objects, related to rhombus tilings via planar duality, are in fact equivalent to *simple wiring diagrams* (a special case of wirings studied in [5]). The latter diagram can be introduced as a set of curves ("wires") $\zeta_1, \dots, \zeta_n$ in the strip $[0, 1] \times \mathbb{R}$ with the following properties: ζ_i begins at the point $(0, i)$ and ends at the point $(1, n - i)$; any two wires intersect at exactly one point; and no three wires have a common point. This is bijective (up to an isotopy) to a rhombus tiling T in which an ij-tile corresponds to the intersection point of wires ζ_i, ζ_j and an i-track corresponds to the wire ζ_i. In their turn, the snakes of T correspond to the so-called *cutpaths* in the wiring (in terminology of [9]).

In light of (i) in Proposition 2, we will not distinguish between snakes S and their corresponding linear orders σ, denoting the snake as $\mathcal{S}(\sigma)$ and saying that the linear order σ is *compatible* with the tiling T. The set of linear orders compatible with T is denoted by $\Sigma(T)$.

Example 1. When $n = 3$, there are exactly two tilings of the zonogon (hexagon) Z_3, as depicted in Fig. 2. Here the set $\Sigma(T)$ consists of four orders, namely: 123, 132, 312 and 321. This is precisely the peak domain $\mathcal{D}(\cap)$. In its turn, the set $\Sigma(T')$ consists of four orders 123, 213, 231 and 321, which is just the pit domain $\mathcal{D}(\cup)$.

So the domains $\Sigma(T)$ and $\Sigma(T')$ in this example are CDs. We will explain later that a similar property holds for any rhombus tiling.

Next, the snakes of a tiling T of the zonogon $Z = Z_n$ are partially ordered "from left to right" in a natural way. The minimal element is the leftmost snake $\mathcal{S}(\alpha) = \mathrm{lbd}(Z)$, and the maximal element is the rightmost snake $\mathcal{S}(\omega) = \mathrm{rbd}(Z)$. The corresponding poset is a (distributive) lattice in which, for two snakes S and S', their greatest lower bound $S \wedge S'$ coincides with their "left envelope", and the least upper bound $S \vee S'$ coincides with the "right envelope". In terms of left regions of snakes (cf. Proposition 2(iii)), we have $L(S \wedge S') = L(S) \cap L(S')$ and $L(S \vee S') = L(S) \cup L(S')$.

Thus, we obtain a natural partial order $\prec$ on the set $\Sigma(T)$ of linear orders, defined by $\sigma \prec \tau \Leftrightarrow L(\sigma) \subset L(\tau)$. Moreover, by (iii) in Proposition 2, the relation $L(\sigma) \subset L(\tau)$ is equivalent to $\mathrm{Inv}(\sigma) \subset \mathrm{Inv}(\tau)$, and therefore the partial order $\prec$ on $\Sigma(T)$ is induced by the weak Bruhat order $\ll$ on $\mathcal{L}([n])$.

In its turn, (iv) in Proposition 2 shows that if a snake $\mathcal{S}(\tau)$ lies on the right from a snake $\mathcal{S}(\sigma)$ and there is no snake between them, then these snakes differ by a single tile. This leads to a sharper version of the above property, namely: *the covering relations on the poset $\Sigma(T)$* (w.r.t. $\prec$) *are induced by covering relations on the Bruhat poset*. As a consequence, we obtain the following

Corollary 1. *Any maximal chain in the poset $\Sigma(T)$ is a maximal chain in the Bruhat poset $(\mathcal{L}, \ll)$.*

C. In the rest of this section, we show that for any rhombus tiling T of Z_n, the set $\Sigma(T)$ is a CD.

We use the track reducing operation defined above. Take the reduction $T' = T|_{[n]-i}$ of T by an alternative i. Then any snake $\mathcal{S}(\sigma)$ compatible with T is transformed into a snake corresponding to the restricted linear order $\sigma|_{[n]-i}$ and compatible with T'. This gives the restriction map

$$\Sigma(T) \to \Sigma(T|_{[n]-i}).$$

Making a sequence of reducing operations, we can reach any subset $X \subset [n]$ and obtain the corresponding restriction map

$$\Sigma(T) \to \Sigma(T|_X).$$

Theorem 1. *The set $\Sigma(T)$ is a complete Condorcet domain.*

Proof. Consider the restrictions of linear orders from $\Sigma(T)$ to a triple ijk. By the reasonings above, they belong to $\Sigma(T|_{ijk})$. The obtained domain is either $\mathcal{D}(\cup)$ or $\mathcal{D}(\cap)$ (defined in Section 3). Therefore, $\Sigma(T)$ is a CD (cf. Proposition 1(i)).

To check the completeness of $\Sigma(T)$, let us try to add to it a new linear order ρ. Then the corresponding path $\mathcal{S}(\rho)$ drawn in Z_n is not contained in G_T. Let e be the *first* edge of $\mathcal{S}(\rho)$ which is not an edge of T, and let v be the beginning vertex of e. Then the part P of $\mathcal{S}(\rho)$ from $b(Z_n)$ to v lies in G_T. Three cases are possible, as depicted in Fig. 3.

Consider the middle case. Let the edge e have color j, and let the tile of T whose interior meets e be an ik-tile Q. Then $i < j < k$. Clearly, the part P of $\mathcal{S}(\rho)$ cannot contain an edge with color in $\{i, j, k\}$. Hence, in the linear order ρ, the alternative j occurs earlier than each of i, k. Two subcases are possible: either $j <_\rho i <_\rho k$ or $j <_\rho k <_\rho i$. In the first subcase, compare ρ with two linear orders from the domain $\Sigma(T)$: a linear order σ that follows the path P and then the left side of Q, yielding the relation $i <_\sigma k <_\sigma j$, and the linear order ω, yielding $k <_\omega j <_\omega i$. This gives a cyclic triple. In the second subcase, act symmetrically, by comparing ρ with a linear order τ that follows P and the right side of Q (yielding $k <_\tau i <_\tau j$) and the linear order α (yielding $i <_\alpha j <_\alpha k$), again obtaining a cyclic triple.

Two other cases are examined in a similar way. □

We refer to a domain of the form $\Sigma(T)$ as a *Condorcet domain of tiling type*, or a *tiling* CD.

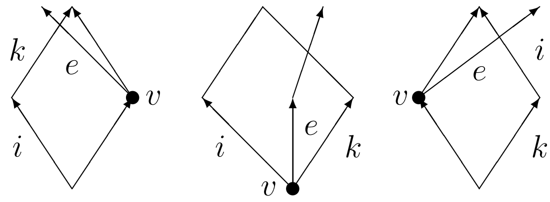

Fig. 3. Three possible dispositions of the edge e.

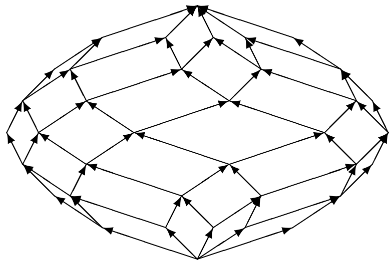

Fig. 4. The tiling for Fishburn's domain with $n = 8$.

5. Tiling CDs and peak–pit domains

A set $\mathcal{D} \subset \mathcal{L}([n])$ is called a *peak–pit domain* if for each triple $i < j < k$ in $[n]$, the peak condition or the pit one is satisfied (in the sense that the projection of $\mathcal{D}$ to $\{i, j, k\}$ is contained either in the peak domain $\mathcal{D}_3(\cap)$ or in the pit domain $\mathcal{D}_3(\cup)$, with ijk in place of 123, or in both). We have the following property (cf. the proof of Theorem 1).

$(*)$ *Any tiling* CD *is a peak–pit domain.*

The converse property is valid as well.

Theorem 2. *Any peak–pit domain is contained in a tiling CD.*

To prove this assertion (which is less trivial), we need some definitions and preliminary observations.

Let $\sigma \in \mathcal{L}([n])$. A subset $X \subseteq [n]$ is called an *ideal* of σ if $x \in X$ and $y <_\sigma x$ imply $y \in X$. In other words, if σ is represented as a word $i_1 \cdots i_n$, then an ideal of σ corresponds to an initial segment of this word. Let $\mathrm{Id}(\sigma)$ denote the set of ideals of σ (including the empty set). In particular, $\mathrm{Id}(\alpha)$ consists of the intervals $[0], [1], \ldots, [n-1], [n]$.

We associate to a collection $\mathcal{D} \subseteq \mathcal{L}([n])$, the following set-system

$$\mathrm{Id}(\mathcal{D}) = \cup_{\sigma\in\mathcal{D}} \mathrm{Id}(\sigma).$$

Example 2. Let $\mathcal{D}$ be the peak domain for $n = 3$; it consists of four orders 123, 132, 312, and 321. Then $\mathrm{Id}(\mathcal{D})$ consists of seven sets $\emptyset$, $\{1\}$, $\{3\}$, $\{1, 2\}$, $\{1, 3\}$, $\{2, 3\}$, and $\{1, 2, 3\} = [3]$, that is, of all subsets of [3] except for $\{2\}$. In its turn, for the pit domain $\mathcal{D}'$, $\mathrm{Id}(\mathcal{D}')$ consists of all subsets of [3] except for $\{1, 3\}$.

Consider a tiling T. We associate to each vertex v in it a certain subset $\mathrm{sp}(v)$ of $[n]$, as follows. Let $\mathcal{S}(\sigma)$ be a snake passing v. Then $\mathrm{sp}(v)$ is the ideal of σ corresponding to the part of $\mathcal{S}(\sigma)$ from the beginning to v (the set $\mathrm{sp}(v)$ does not depend on the choice of a snake σ passing v). This is equivalent to saying that $\mathrm{sp}(v)$ consists of the elements $i \in [n]$ such that the i-track goes below the vertex v (in view of Proposition 2(i)). The collection of sets $\mathrm{sp}(v)$ for all vertices v of T is denoted by $\mathrm{Sp}(T)$ and called the *spectrum* of T (following terminology in [5]). One can check that a linear order σ belongs to $\Sigma(T)$ if and only if the inclusion $\mathrm{Id}(\sigma) \subset \mathrm{Sp}(T)$ holds.

Proof of Theorem 2. Let $\mathcal{D} \subset \mathcal{L}([n])$ be a peak–pit domain. Our aim is to show the existence of a tiling T such that $\mathrm{Id}(\mathcal{D}) \subseteq \mathrm{Sp}(T)$. We use a criterion due to Leclerc and Zelevinsky [14] on a system of subsets of $[n]$ that can be extended to the spectrum $\mathrm{Sp}(T)$ of a tiling T. (Strictly speaking, the criterion in [14] concerns set-systems associated with pseudo-line arrangements, which correspond, in a sense, to rhombus tilings; cf. [7]. For a direct proof, in terms of tilings, see [4, Sec. 5.3].)

Two subsets A, B of $[n]$ are said to be *separated* (more precisely, *strongly separated*, in terminology of [14]) from each other if the convex hulls of $A \setminus B$ and $B \setminus A$ (viz. the minimal intervals containing these sets) are disjoint. For example, the sets $\{1, 2\}$ and $\{2, 4\}$ are separated, whereas $\{1, 3\}$ and $\{2\}$ are not. In particular, A and B are separated if one includes the other. A collection of sets is called *separated* if any two sets in it are separated.

Theorem 3 (*[14]*)**.** *The spectrum* $\mathrm{Sp}(T)$ *of any rhombus tiling T is separated. Conversely, if $\mathcal{X}$ is a separated set-system on $[n]$, then there exists a tiling T of Z_n such that $\mathcal{X} \subset \mathrm{Sp}(T)$.*

Due to this theorem, it suffices to show that for a peak–pit domain $\mathcal{D}$, the system $\mathrm{Id}(\mathcal{D})$ is separated. Suppose that this is not so for some $\mathcal{D}$. Then there are two sets $A, B \in \mathrm{Id}(\mathcal{D})$ and a triple $i < j < k$ in $[n]$ such that A contains j but none of i, k, whereas B contains i, k but not j. Restrict the members of $\mathcal{D}$ to the set $\{i, j, k\}$. Then $\mathrm{Id}(\mathcal{D}|_{ijk})$ contains both sets $\{j\}$ and $\{i, k\}$. Thus, we are neither in the peak nor in the pit domain case, as we have seen in Example 2. □

Now we combine Theorem 2 and a slight modification of property $(*)$ (in the beginning of this section), yielding the main assertion in this paper. Let us say that a domain $\mathcal{D}$ is *semi-connected* if the linear orders α and ω can be connected in the Bruhat graph by a path in which all vertices belong to $\mathcal{D}$.

Theorem 4. (i) *Every domain of tiling type is semi-connected.*
(ii) *Every semi-connected Condorcet domain is a peak–pit domain.*
(iii) *Every peak–pit domain is contained in a domain of tiling type.*

Proof. Any domain of the form $\Sigma(T)$ is semi-connected since it contains a maximal chain of the Bruhat poset (cf. Corollary 1), yielding (i).

It is easy to see that the semi-connectedness preserves under reducing alternatives. Because of this, we can restrict ourselves to the case $n = 3$. In this case, there exist exactly four CDs. Two of them, where one of the alternatives 1 and 3 is never the middle, are not semi-connected. The other two domains are semi-connected; they are just the peak and pit domains. This implies (ii).

Claim (iii) is just Theorem 2. □

As a consequence, we obtain that the CDs constructed by Abello [1], Chameni-Nembua [3], and Galambos and Reiner [9] (see Appendix for a brief outline), as well as the maximal peak–pit domains, are CDs of tiling type. Moreover, all these classes of CDs are equal.

6. On Fishburn's conjecture

Fishburn [8] constructed Condorcet domains by the following method. For a set of linear orders and a triple $i < j < k$, Fishburn's "never condition" $jN1$ means the requirement that, in the restriction of each of these linear orders to $\{i, j, k\}$, the alternative j is never the worst. This is exactly the above-mentioned "peak condition" for ijk. Similarly, the "never condition" $jN3$ (saying that "the alternative j is never the best") coincides with the "pit condition" for ijk.

Fishburn's *alternating scheme* is defined by imposing, for each triple $i < j < k$, the peak condition when j is even, and the pit condition when j is odd. The set of linear orders (individually) obeying these conditions is called *Fishburn's domain*, and its cardinality is denoted by $\Phi(n)$.

By Theorem 2, Fishburn's domain $\mathcal{D}$ is contained in a CD of tiling type. Also it is a complete CD, as is shown in [9]. So $\mathcal{D}$ is a tiling CD. The corresponding tiling for $n = 8$ is drawn in Fig. 4.

Fishburn conjectured that *the size of any peak–pit* CD *does not exceed* $\Phi(n)$, and verified this conjecture for $n \le 6$.

Galambos and Reiner [9] considered a class of CDs, which we call *GR-domains* (see the definition in Appendix), and raised a weakened version of Fishburn's conjecture saying that *the size of any* GR-*domain does not exceed* $\Phi(n)$. It should be noted that an equivalent conjecture in terms of pseudo-line arrangements was raised earlier by Knuth [13].

Monjardet [15] calls a CD *connected* if it induces a connected subgraph of the Bruhat graph. He conjectured that *the size of any connected CD does not exceed* $\Phi(n)$.

Applying Theorem 4, one can conclude that the conjectures by Fishburn, Galambos and Reiner, and Monjardet are equivalent, and we can express this conjecture as follows:

(C) *the maximum possible size* γ_n *of a tiling CD for* n *is equal to* $\Phi(n)$.

However, (C) is not true in general. The authors learnt via Monjardet (however, without pointing out to us any details or references) that Ondjey Bilka had established some lower bound on γ_n which leads to a contradiction with (C). Subsequently, the authors found a simple argument, as follows.

Let T and T' be rhombus tilings of zonogons Z_n and $Z_{n'}$, respectively. We identify the set $[n']$ with the subset $\{n+1, \ldots, n+n'\}$ in $[n+n']$ and merge the top vertex $t(T)$ of T with the bottom vertex $b(T')$ of T' (erecting T' over T). This gives a "partial tiling" of the zonogon $Z_{n+n'}$, as illustrated in Fig. 5 where $n = 4$ and $n' = 3$.

This partial tiling can be extended (in a unique way, in fact) to a complete rhombus tiling $\widehat{T}$ of the whole zonogon $Z_{n+n'}$. If σ is a snake of T and σ' is a snake of T', then the concatenated path $\sigma\sigma'$ is a snake of $\widehat{T}$. Thus, we obtain the injective mapping

$$\Sigma(T) \times \Sigma(T') \to \Sigma(\widehat{T}),$$

which gives the inequality $\gamma_n \gamma_{n'} \le \gamma_{n+n'}$.

Now take both T and T' to be Fishburn's tilings for $n = n' = 21$. Using a precise formula for $\Phi(n)$ from [9], one can compute that $\Phi(21) = 4.443.896$ and $\Phi(42) = 19.156.227.207.750$. Then $\Phi(21)^2 = 19.748.211.658.816 > \Phi(42)$. Thus, $\Phi(42) < \gamma_{42}$, contradicting (C).

Remark. The above construction can be given in terms of "concatenating" corresponding peak–pit domains rather than tilings. So Fishburn's conjecture can be disproved without appealing to Theorem 4.

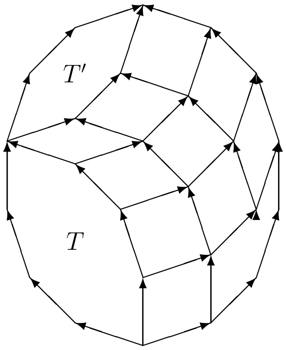

Fig. 5. "Concatenation" of tilings T and T'.

7. Some reformulations

Any linear order can be realized as a snake of some rhombus tiling. However, this need not hold for a pair of linear orders. For example, the linear orders 213 and 312 (which together with 123 and 321 form the CD $\mathcal{D}_3(\leftarrow)$ from Section 3) cannot appear in the same tiling.

Let us say that two linear orders σ and τ are *strongly consistent* if there exists a tiling T such that $\sigma, \tau \in \Sigma(T)$. For example, σ and τ are strongly consistent if $\sigma \ll \tau$ (where $\ll$ is defined in Section 2). Using observations and results from previous sections, we can demonstrate some useful equivalence relations.

Proposition 3. *Let σ and τ be linear orders on $[n]$. The following properties are equivalent:*

(i) *σ and τ are strongly consistent;*
(ii) *the set-system $\mathrm{Id}(\sigma) \cup \mathrm{Id}(\tau)$ is separated;*
(iii) *for each triple in $[n]$, the restrictions of σ and τ to this triple simultaneously satisfy either peak conditions or pit conditions (or both);*
(iv) *$\mathrm{Id}(\sigma) \cup \mathrm{Id}(\tau) = \mathrm{Id}(\sigma \vee \tau) \cup \mathrm{Id}(\sigma \wedge \tau)$ (where $\vee, \wedge$ concern the Bruhat lattice);*
(iv′) *$\mathrm{Id}(\sigma) \cup \mathrm{Id}(\tau) \subseteq \mathrm{Id}(\sigma \vee \tau) \cup \mathrm{Id}(\sigma \wedge \tau)$.*

Proof. Properties (i) and (ii) are equivalent by Theorem 3.

Properties (i) and (iii) are equivalent by Theorem 2.

To see that (i) implies (iv), observe that if σ and τ occur in a tiling T; then $\mathcal{S}(\sigma \vee \tau)$ and $\mathcal{S}(\sigma \wedge \tau)$ are the left and right envelopes of the snakes for σ and τ, respectively. Therefore, any vertex of the snake $\mathcal{S}(\sigma \vee \tau)$ is a vertex of $\mathcal{S}(\sigma)$ or $\mathcal{S}(\tau)$, and similarly for $\mathcal{S}(\sigma \wedge \tau)$. Conversely, each vertex of $\mathcal{S}(\sigma) \cup \mathcal{S}(\tau)$ is a vertex of $\mathcal{S}(\sigma \vee \tau)$ or $\mathcal{S}(\sigma \wedge \tau)$.

Obviously, (iv) implies (iv′). Let us prove the converse. Since $\sigma \wedge \tau \ll \sigma \vee \tau$, the linear orders $\sigma \wedge \tau$ and $\sigma \vee \tau$ are strongly consistent. By the equivalence of (i) and (ii), $\mathrm{Id}(\sigma \vee \tau) \cup \mathrm{Id}(\sigma \wedge \tau)$ is a separated system. Since $\mathrm{Id}(\sigma) \cup \mathrm{Id}(\tau) \subseteq \mathrm{Id}(\sigma \vee \tau) \cup \mathrm{Id}(\sigma \wedge \tau)$, the set-system $\mathrm{Id}(\sigma) \cup \mathrm{Id}(\tau)$ is separated as well. Thus, we obtain (ii), whence (iv′) ⇒ (iv). □

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Appendix

Here we briefly outline approaches of Abello [1], Galambos and Reiner [9], and Chameni-Nembua [3], and an interrelation between theirs and our approach.

Abello

Let $\mathcal{D}$ be a CD. Then there exists a casting c such that $\mathcal{D} \subseteq \mathcal{D}(c)$ (see Proposition 1). Abello applied this fact to a maximal chain $\mathcal{C}$ in the Bruhat lattice (it had been known that any chain is a CD). In this case, the casting c is unique (and is a peak–pit casting); so the domain $\mathcal{D}(c)$, denoted by $\widehat{\mathcal{C}}$, is a CD as well. We call such a CD an *A-domain* (abbreviating *Abello's domain*). Abello shows that an A-domain is a complete CD.

Note that different chains can give the same A-domain. Maximal chains $\mathcal{C}$ and $\mathcal{C}'$ are called *equivalent* if the A-domains $\widehat{\mathcal{C}}$ and $\widehat{\mathcal{C}'}$ coincide. In the end of [1], Abello gives another characterization of this equivalence. A maximal chain in the Bruhat lattice can be thought of as a reduced decomposition (a product of standard transpositions s_i, $i \in [n-1]$) of the longest permutation ω. Then two chains are equivalent if one reduced decomposition can be obtained from the other by a sequence of transformations, each replacing a decomposition fragment of the form $s_i s_j$ with $|i-j| > 1$ by $s_j s_i$. This characterization became a starting point in Galambos and Reiner's approach.

Galambos and Reiner

Let $\mathbf{C}$ be an equivalence class of maximal chains in the Bruhat lattice. Define $\mathcal{D}(\mathbf{C}) := \cup_{\mathcal{C}\in\mathbf{C}} \mathcal{C}$ (Galambos and Reiner referred to this domain as consisting of "permutations visited by an equivalence class of maximal reduced decompositions"). We call $\mathcal{D}(\mathbf{C})$ a GR-*domain*. It is easy to see (and Galambos and Reiner explicitly mention this) that the GR-domains are exactly the A-domains. Moreover, they give a direct proof (in Theorems 1 and 2 of [9]) that a GR-domain is a complete CD.

To give a more enlightening representation for the equivalence classes of maximal reduced decompositions, Galambos and Reiner used *arrangements of pseudo-lines* (cf. [2]). Permutations (or linear orders) from the domain $\mathcal{D}(\mathbf{C})$ are realized in these arrangements as certain cutpaths (viz. directed cuts). Although they do not prove explicitly that the set of cutpaths of an arrangement forms a complete CD, this can be done rather easily. Using a relationship between pseudo-line arrangements and rhombus tilings (cf. [7]), one can conclude that the GR-domains (as well as the A-domains) are exactly CDs of tiling type.

Chameni-Nembua

One more interesting approach was proposed by Chameni-Nembua. A sublattice $\mathcal{D}$ in the Bruhat lattice is called *covering* if the cover relation in this sublattice is induced by the cover relation in the Bruhat lattice.

Chameni-Nembua shows that a distributive covering sublattice in the Bruhat lattice is a CD. Suppose that $\mathcal{D}$ is a maximal distributive covering sublattice. One can easily see that it contains α and ω, and hence it contains a maximal chain. Therefore, it is a subset of a unique tiling CD. On the other hand, since any tiling CD forms a distributive covering sublattice (see Section 4), one can conclude that $\mathcal{D}$ coincides with this tiling CD.

Thus, Chameni-Nembua's approach gives the same class of CDs as the one of rhombus tilings.

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